INVESTIGATION OF A PIECEWISE LINEAR DYNAMICAL SYSTEM WITH THREE PARAMETERS

(ISSLEDOVANIE ODNOI KUSOCHNO-LINEINOI DINAMICHESKOI SISTEMY S TREMIA PARAMETRAMI)

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A qualitative investigation was carried out of a cylindrical phase space for one piecewise linear system which is of interest in applications. In particular, an existence of a semistable limit cycle encompassing the cylinder was established. Obtained are analytic expressions for all bifurcate surfaces dividing the parameter space into regions of equal qualitative structure.

1. Statement of problem. The equation

$$\ddot{x} + \alpha\,[1 - \beta F'(x)]\,\dot{x} + F(x) = \gamma$$

where $F(x)$ is a periodic function with period 2π, results from a consideration of a number of electromechanical and mechanical systems (alternating current synchronous machines, automatic control systems, pendulum theory, etc.). This equation has been considered with various approximations of the function $F(x)$. The conditions for generation of a limit cycle from a separatrix passing from a saddle to saddle for piecewise linear approximation of $F(x)$ and $\alpha > 0$, $\beta < 1$ are found in [1]. Reference [2] gives, for sinusoidal approximation of $F(x)$ and $\alpha \geqslant 0$ $\beta \geqslant 0$, a qualitative investigation of the equation and evaluates locations of the bifurcate surface for the separatrix passing from a saddle to saddle in the parameter space. Voluminous literature is devoted to the particular case $\beta = 0$ (see for example [3-8]).

A qualitative investigation and analytic expressions for all bifurcate surfaces are given in the proposed work for piecewise linear approximation of the function $F(x)$ and arbitrary values of α and β. Assuming

$$F(x) = \frac{2}{\pi}x \quad \text{for } -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}, \qquad F(x) = -\frac{2}{\pi}x + 2 \quad \text{for } \frac{\pi}{2} \leqslant x \leqslant \frac{3\pi}{2}$$

and introducing new variables and parameters we obtain a system of the type

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -x - 2h_1 y + a \quad \text{for} \ -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2} \qquad (1.1)$$

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = x - 2h_2 y - (\pi - a) \quad \text{for} \ \frac{\pi}{2} \leqslant x \leqslant \frac{3\pi}{2} \qquad (1.2)$$

We will consider a cylindrical phase surface of the joint system (1.1)-(1.2) developed on part of a surface corresponding to the inequalities $-\pi/2 \leqslant x \leqslant 3\pi/2$. The straight line $x = \pi/2$ divides the considered surface into regions (1) and (2) in each of which the phase trajectories are determined by the linear systems (1.1) and (1.2), respectively. The lines $x = -\pi/2$ and $x = 3\pi/2$ are identified in Fig. 1. The joint system (1.1)-(1.2) has two states of equilibrium $O_1(a, 0)$ and $O_2(\pi - a, 0)$. The point O_1 is a simple critical point of the system (1.1). Obviously, O_1 will be a stable focus for $0 < h_1 < 1$, an unstable one for $-1 < h_1 < 0$, and a center for $h_1 = 0$. Furthermore, the critical point O_1 is a stable node for $h_1 > 1$, an unstable node for $h_1 < -1$ and a dicritical node for $|h_1| = 1$. Point O_2 is a simple critical point of the system (1.2). For all values of the parameter h_2, the point O_2 is a saddle whose separatrices are defined by the equations

$$y = (-h_2 \pm \sqrt{h_2^2 + 1})\,(x - (\pi - a))$$

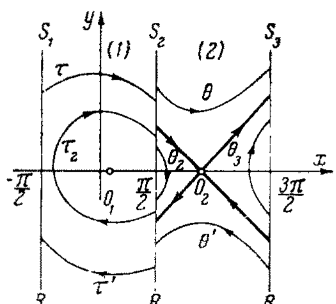

Fig. 1.

The existence of limit cycles for the system (1.1)-(1.2) can be established by considering the corresponding point transformations. Indeed, if a limit cycle exists which does not encompass the cylinder, then it must contain within itself the point O_1 and either intersect one line $x = \pi/2$ or intersect both lines $x = \pi/2$ and $x = 3\pi/2$. Thus, in order to determine limit cycles not encompassing the cylinder it is necessary to consider the point transformation of the line $x = \pi/2$ into itself, as well as a more complicated transformation for the case when the limit cycle intersects two straight lines. In order to determine limit cycles encompassing the cylinder it is necessary to consider point transformations of the lines $x = -\pi/2$, $x = \pi/2$ and $x = 3\pi/2$ into each other.

Let us denote by S_1, S_2, S_3 (R_1, R_2, R_3) the lines $x = -\pi/2$, $x = \pi/2$, $x = 3\pi/2$ corresponding to $y > 0$ ($y < 0$). Phase trajectories of the system (1.1) effect the point transformations of S_1 into S_2, R_2 into R_1 and R_2 into S_2. Let us denote these transformations, respectively, by

$L^{(1)}$, $L_-^{(1)}$ and $\Pi_2^{(1)}$. Phase trajectories of the system (1.2) effect the point transformations of S_2 into S_3, R_3 into R_2, S_2 into R_2 and R_3 into S_3. These transformations will be denoted, respectively, by $L^{(2)}$, $L_-^{(2)}$, $\Pi_2^{(2)}$ and $\Pi_3^{(2)}$ (Fig. 1).

Subsequently we will use the following notation:

$$\omega_1 = \sqrt{1 - h_1^2} \quad \text{for } |h_1| < 1, \qquad \omega_1 = \sqrt{h_1^2 - 1} \quad \text{for } |h_1| > 1$$

$$k_1 = \frac{h_1}{\omega_1}, \qquad \omega_2 = \sqrt{h_2^2 + 1}, \qquad k_2 = \frac{h_2}{\omega_2}$$

Let s_1, s_2, s_3, r_1, r_2, r_3 be the ordinates of the respective straight line points; τ/ω_1, τ'/ω_1, θ/ω_2, θ'/ω_2, τ_2/ω_1, θ_2/ω_2, θ_3/ω_2 duration times for the describing point (Fig. 1) to pass through the regions corresponding to transformations $L^{(1)}$, $L_-^{(1)}$, $L^{(2)}$, $L_-^{(2)}$, $\Pi_2^{(1)}$, $\Pi_2^{(2)}$, $\Pi_3^{(2)}$. Quantities s_1, ..., r_3, as well as τ, τ', ..., θ_3 assume positive or zero values.

Let us derive the equations in parametric form of correspondence functions (see for example [4, Chapt. 3]) for each of the indicated transformations

Transformations $L^{(1)}$ and $L_-^{(1)}$

$$s_1(\tau, h_1, a) = \left(\frac{\pi}{2} - a\right)\frac{\omega_1}{\sin\tau} e^{k_1\tau} + \left(\frac{\pi}{2} + a\right)(\omega_1 \cot\tau + h_1)$$

$$s_2(\tau, h_1, a) = s_1(\tau, -h_1, -a) \tag{1.3}$$

$$\frac{ds_1}{ds_2} = e^{2k_1\tau}\frac{s_2}{s_1} \tag{1.4}$$

$$\frac{d^2s_1}{ds_2^2} = \frac{\sin\tau}{\omega_1 s_1^3} e^{3k_1\tau}\left[\left(\frac{\pi}{2} - a\right)s_1 - \left(\frac{\pi}{2} + a\right)s_2\right] \tag{1.5}$$

In case $|h_1| > 1$, the expressions for $s_1(\tau, h_1, a)$, $s_2(\tau, h_1, a)$ and derivatives are obtained if on the right-hand sides of the equalities (1.3) and (1.5) one substitutes $\sinh\tau$ and $\coth(\tau)$ for $\sin\tau$ and $\cot\tau$. The equality (1.4) remains unchanged.

In cases $h_1 = \pm 1$, one needs to replace τ by $\omega_1\tau$ in the equalities (1.3) to (1.5) and pass to the limit for $h_1 \to \pm 1$.

It follows from Expression (1.3) and the corresponding phase plot that for $|h_1| < 1$ the following cases are possible.

1) If $(\pi/2 - a)e^{k_1\pi} > (\pi/2 + a)$, then in order to obtain all possible

values of s_1 and s_2 the parameter τ should be varied from zero to some value $\tau = \tau^* < \pi$ for which $s_2(\tau^*) = 0$, $s_1(\tau^*) > 0$.

2) If $(\pi/2 - a)e^{k_1\pi} = (\pi/2 + a)$, then $0 < \tau \leqslant \pi$ and $s_1(\pi) = s_2(\pi) = 0$.

3) If $(\pi/2 - a)e^{k_1\pi} < (\pi/2 + a)$, then $0 < \tau < \tau_0 < \pi$ and $s_1(\tau_0) = 0$, $s_2(\tau_0) > 0$.

In case when the point O_1 is a node, i.e. for $|h_1| \geqslant 1$, the parameter τ should be varied from zero to some value $\tau = \tau^* > 0$ for which $s_2(\tau^*) = 0$, $s_1(\tau^*) > 0$ if $h_1 > 0$; the parameter τ is varied from zero to some value τ_0 for which $s_1(\tau_0) = 0$, $s_2(\tau_0) > 0$ if $h_1 < 0$.

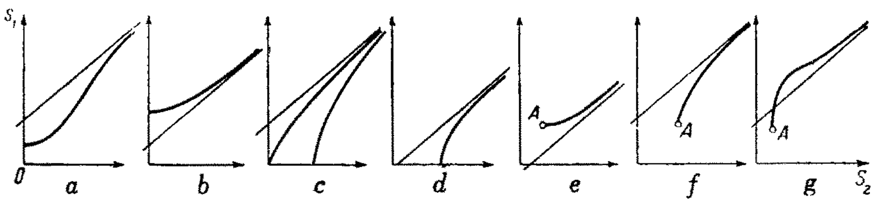

Fig. 2.

Equations (1.3) and the corresponding equations for $|h_1| \geqslant 1$ define for the transformation $L^{(1)}$ the correspondence function $s_1 = s_1(s_2)$. For all values of the parameters a and h_1 the curve $s_1 = s_1(s_2)$ has the asymptote

$$s_1 = s_2 + 2\pi h_1 \tag{1.6}$$

The equalities (1.3) to (1.6) allow one to determine the form of the curve $s_1 = s_1(s_2)$. It is shown in Fig. 2 where

$$0 < a < \frac{\pi}{2}, \ h_1 \geqslant 1 \quad \text{or} \quad 0 < h_1 < 1, \ \left(\frac{\pi}{2} - a\right)e^{k_1\pi} > \left(\frac{\pi}{2} + a\right) \quad \text{(Fig. 2}a\text{)}$$

$$a = 0, \qquad h_1 > 0 \quad \text{(Fig. 2}b\text{)}$$

$$0 < a < \pi/2, \quad 0 \leqslant h_1 < 1, \quad (\pi/2 - a)\, e^{k_1\pi} \leqslant (\pi/2 + a) \quad \text{(Fig. 2}c\text{)}$$

$$0 \leqslant a < \pi/2, \qquad h_1 < 0 \quad \text{(Fig. 2}d\text{)}$$

The equations for the correspondence function of $L_-^{(1)}$ transformation are obtained if in the equalities (1.3) and the corresponding equalities for $|h_1| \geqslant 1$ one assumes $s_1 \leqslant 0$, $s_2 \leqslant 0$, $\tau \leqslant 0$ and subsequently replace s_1, s_2 and τ by $-r_1$, $-r_2$, $-\tau'$, respectively. It is easy to see that the following equalities are satisfied:

$$r_1(\tau', h_1) = s_1(\tau', -h_1), \qquad r_2(\tau', h_1) = s_2(\tau', -h_1) \tag{1.7}$$

Transformations $L^{(2)}$ and $L_-^{(2)}$

$$s_3(\theta, h_2, a) = \left(\frac{\pi}{2} - a\right)\frac{\omega_2}{\sinh\theta} e^{-k_2\theta} + \left(\frac{\pi}{2} + a\right)(\omega_2 \coth\theta - h_2)$$

$$s_2(\theta, h_2, a) = s_3(\theta, -h_2, -a) \tag{1.8}$$

$$\frac{ds_3}{ds_2} = e^{-2k_2\theta}\frac{s_2}{s_3} \tag{1.9}$$

$$\frac{d^2s_3}{ds_2^2} = -\frac{\sinh\theta}{\omega_2 s_3^3} e^{-3k_2\theta}\left[\left(\frac{\pi}{2} - a\right)s_3 - \left(\frac{\pi}{2} + a\right)s_2\right] \tag{1.10}$$

It follows from Expressions (1.8) and the corresponding phase plot that the parameter θ should be varied from zero to infinity. Then s_2 and s_3 tend to ∞ for $\theta \to 0$ and

$$\lim s_3 = (\pi/2 + a)(-h_2 + \omega_2), \quad \lim s_2 = (\pi/2 - a)(h_2 + \omega_2) \quad \text{for } \theta \to \infty$$

Equations (1.8) define the correspondence function $s_3 = s_3(s_2)$ for the transformation $L^{(2)}$. It is easy to see that this function has the asymptote

$$s_3 = s_2 - 2\pi h_2 \tag{1.11}$$

The point $A[(h_2 + \omega_2)(\pi/2 - a), (-h_2 + \omega_2)(\pi/2 + a)]$ on the curve $s_3 = s_3(s_2)$ corresponds to $\theta = \infty$. The form of this curve, determined with the aid of the equalities (1.8) to (1.11), is shown in Fig. 2:

$$0 < a < \pi/2, \quad h_2 \geqslant 0 \quad \text{or} \quad a = 0, \quad h_2 > 0 \quad \text{(Fig. 2}e\text{)}$$

$$a = 0, \quad h_2 < 0 \quad \text{(Fig. 2}f\text{)}$$

$$0 < a < \pi/2, \quad h_2 < 0 \quad \text{(Fig. 2}g\text{)}$$

By similar reasoning as in the case for $L_-^{(1)}$ transformation we get that for the correspondence function of $L_-^{(2)}$ transformation, which is defined by the equalities $r_2 = r_2(\theta')$ and $r_3 = r_3(\theta')$, the following relations are satisfied:

$$r_3(\theta', h_2) = s_3(\theta', -h_2), \quad r_2(\theta', h_2) = s_2(\theta', -h_2) \tag{1.12}$$

Transformation $\Pi_2^{(1)}$

$$s_2(\tau_2, h_1) = \left(-\frac{\omega_1}{\sin\tau_2} e^{-k_1\tau_2} + \omega_1 \cot\tau_2 - h_1\right)(\pi/2 - a)$$

$$r_2(\tau_2, h_1) = s_2(\tau_2, -h_1) \tag{1.13}$$

$$\frac{ds_2}{dr_2} = e^{-2k_1\tau_2}\frac{r_2}{s_2} \tag{1.14}$$

Here $|h_1| < 1$. The parameter τ_2 should be varied from π to some value $\tau_2 = \tau_2^{\circ} \leqslant 2\pi$ for which

$$s_2(\tau_2^\circ) = 0, \quad r_2(\tau_2^\circ) > 0, \quad \text{if} \quad h_1 > 0$$
$$r_2(\tau_2^\circ) = 0, \quad s(\tau_2^\circ) > 0 \quad \text{if} \quad h_1 < 0$$
$$r_2(\tau_2^\circ) = 0, \quad s_2(\tau_2^\circ) = 0, \quad \text{if} \quad h_1 = 0$$

In order to obtain the corresponding formulas for transformations $\Pi_2^{(2)}$ and $\Pi_3^{(2)}$, we introduce the notation

$$u(\theta, h_2) = -\omega_2\left(\frac{e^{k_2\theta}}{\operatorname{sh}\theta} - \coth\theta\right) + h_2, \qquad v(\theta, h_2) = u(\theta, -h_2) \tag{1.15}$$

It is easy to see that u and v tend to zero for $\theta \to 0$ and lim $u = (h_2 + \omega_2)$, lim $v = (-h_2 + \omega_2)$ for $\theta \to \infty$. Furthermore

$$\frac{du}{d\theta} = \frac{\omega_2}{\sinh\theta} e^{k_2\theta} v, \qquad \frac{dv}{d\theta} = \frac{\omega_2}{\sinh\theta} e^{-k_2\theta} u \tag{1.16}$$

$$\frac{du}{dv} = e^{2k_2\theta}\frac{v}{u} \tag{1.17}$$

It is easy to show that for $h_2 > 0$ and $h_2 < 0$ the following inequalities are satisfied respectively:

$$u(\theta, h_2) - v(\theta, h_2) > 0, \qquad u(\theta, h_2) - v(\theta, h_2) < 0 \tag{1.18}$$

One can show also that for all positive values of the quantities h_1, h_2 and θ the inequality

$$[v^2(\theta) + 2v(\theta) h_1 + 1] e^{2k_2\theta} > [u^2(\theta) - 2u(\theta) h_1 + 1] \tag{1.19}$$

is satisfied.

Transformations $\Pi_2^{(2)}$ *and* $\Pi_3^{(2)}$

$$s_2(\theta_2) = u(\theta_2)(\pi/2 - a), \qquad r_2(\theta_2) = v(\theta_2)(\pi/2 - a) \tag{1.20}$$

$$s_3(\theta_3) = v(\theta_3)(\pi/2 + a), \qquad r_3(\theta_3) = u(\theta_3)(\pi/2 + a) \tag{1.21}$$

All conclusions about correspondence functions and their derivatives for transformations $\Pi_2^{(2)}$ and $\Pi_3^{(2)}$ follow from Formulas (1.15) to (1.18), (1.20) to (1.21).

2. Limit cycles encompassing the cylinder. Let us consider the complex transformation $L = L^{(1)}L^{(2)}$. To the immovable points of this transformation correspond the limit cycles encompassing the cylinder which are located in the upper part of the phase surface $y > 0$. In order to find these immovable points it is necessary to find the intersections of the curves $s_1 = s_1(s_2)$ and $s_3 = s_3(s_2)$.

Utilizing the relations (1.4) to (1.6), (1.9) to (1.11), one can establish the presence and quantity of the intersections of the curves and the character of the corresponding limit cycles based on the following easily proved propositions.

1. If at all points of intersection the inequality $d(s_1 - s_3)/ds_2 < 0$ (> 0) is satisfied, then there can be no more than one intersection which, on the strength of a Koenings theorem ([4, Chapt. 5]), corresponds to a stable (respectively unstable) limit cycle.

2. If $d^2(s_1 - s_3)/ds_2{}^2 < 0$ (> 0) for all values of s_2 and the difference $s_1 - s_3 < 0$ (> 0) for sufficiently large values of s_2, then there can be no more than one point of intersection and this point corresponds to the stable (unstable) limit cycle.

It is easy to see that bifurcations take place for which the points of intersection for $s_1 = s_1(s_2)$ and $s_3 = s_3(s_2)$ appear or disappear:

a) if point A of curve $s_3 = s_3(s_2)$ lies on the curve $s_1 = s_1(s_2)$;

b) if the asymptotes of the considered curves coincide;

c) if the considered curves are touching.

Case (a). The equalities

$$s_1(\tau) = (-h_2 + \omega_2)(\pi/2 + a), \qquad s_2(\tau) = (h_2 + \omega_2)(\pi/2 - a)$$

must be satisfied where $s_1(\tau)$ and $s_2(\tau)$ are defined by the equalities (1.3) for $|h_1| < 1$ and the corresponding equalities for $|h_1| \geqslant 1$. Eliminating the parameter τ and introducing the notation

$$b = \ln\frac{\pi/2 + a}{\pi/2 - a} \tag{2.1}$$

we obtain the following relationship between the parameters a, h_1 and h_2 for which Case (a) is realized:

$$b = \frac{1}{2}\ln\frac{(\omega_2 + h_2)(\omega_2 + h_1)}{(\omega_2 - h_2)(\omega_2 - h_1)} + D(h_1, h_2) \tag{2.2}$$

Here

$$D(h_1, h_2) = \begin{cases} k_1\left(\pi - \tan^{-1}\dfrac{\omega_1\omega_2}{h_1h_2}\right) & \text{for } 0 \leqslant h_1 < 1, \ h_2 \geqslant 0 \\ k_1\tan^{-1}\dfrac{\omega_1\omega_2}{-h_1h_2} & \text{for } h_1h_2 \leqslant 0, \ |h_1| < 1 \\ k_1\tanh^{-1}\dfrac{\omega_1\omega_2}{-h_1h_2} & \text{for } h_1h_2 \leqslant 0, \ |h_1| > 1 \\ \pm\dfrac{\omega_2}{h_2} & \text{for } h_1h_2 \leqslant 0, \ h_1 = \pm 1 \end{cases} \tag{2.3}$$

The values of the parameters a, h_1 and h_2, satisfying the equalities

(2.2) to (2.3), will be bifurcate. For these values the separatrix passes from saddle to saddle in the upper phase half-plane encompassing the cylinder (Fig. 3). Considering the behavior of the curves $s_1 = s_1(s_2)$ and $s_3 = s_3(s_2)$ one may conclude that for $h_2 \geqslant 0$ ($h_2 < 0$) and with increase (decrease) of parameter a or with decrease (increase) of parameter h_2 in the upper phase half-plane the separatrix passing from a saddle to saddle generates one stable (unstable) limit cycle.

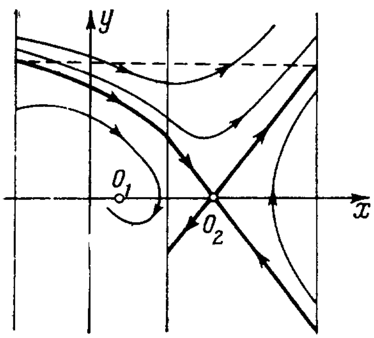

Fig. 3.

Case (b). This case is characterized by the equality

$$h_1 + h_2 = 0 \tag{2.4}$$

In this case the limit cycle appears from infinity on the phase plane. Namely, for values of the parameters $a \geqslant 0$, $h_1 < 0$ and $a > 0$, $h_1 > 0$ ($a = 0$, $h_1 > 0$) and in passing from the inequality $h_1 + h_2 < 0$ ($h_1 + h_2 > 0$) to inequality $h_1 + h_2 > 0$ ($h_1 + h_2 < 0$) there appears from infinity one stable (unstable) limit cycle encompassing the cylinder in the upper phase half-plane.

Case (c). It can be shown that this case is realized only for values of the parameters h_1 and h_2 satisfying the inequalities

$$h_1 > 0, \quad h_2 < 0, \quad h_1 + h_2 > 0, \quad h_1^2 - h_2^2 < 1 \tag{2.5}$$

On the strength of the relations (1.4) and (1.9) Case (c) is characterized by the equalities $e^{2k_1\tau} = e^{-2k_2\theta}$ and $s_1(\tau) = s_3(\theta)$, where $s_1(\tau)$ is defined in the same way as in Case (a), while $s_3(\theta)$ is defined by the equality (1.8).

Eliminating parameters τ and θ and utilizing the notation (2.1), we obtain the relationship between the parameters a, h_1 and h_2:

$$b = k_1\tau_0 - \tfrac{1}{2}\ln \varphi_2(\tau_0) \tag{2.6}$$

where $\tau_0 > 0$ is the root of equation

$$\exp(-k_1\tau / k_2) = \varphi_1(\tau) \tag{2.7}$$

If $h_1 < 1$, then

$$\varphi_1(\tau) = \frac{\sin\tau_1 + \sin\tau}{\tau_1 - \tau}, \quad \varphi_2(\tau) = \frac{\cos(\tau - \tau_2) - 1}{\cos(\tau + \tau_2) + 1}$$

$$\tau_1 = \tan^{-1}\frac{\omega_1\omega_2}{-h_1h_2}, \quad \tau_2 = \tan^{-1}\frac{\omega_1(h_1 + h_2)}{1 - h_1^2 - h_1h_2}$$

If $h_1 > 1$, then

$$\varphi_1(\tau) = \frac{e^{\tau+\tau_1} - 1}{e^{\tau_1} - e^{\tau}}$$

while the expressions for $\phi(\tau)$, τ_1 and τ_2 will be obtained if in the corresponding expressions for the case $h_1 < 1$ one substitutes the hyperbolic cosine and arc tangent for the cosine and the arc tangent of the same arguments.

For $h_1 = 1$ we get

$$b = \tau_0 - \ln \varphi_2(\tau_0) \tag{2.8}$$

Here $\tau_0 > 0$ is the root of equation $\exp(-\tau/k_2) = \phi_1(\tau)$ and

$$\varphi_1(\tau) = \frac{-k_2\tau + 1}{+1 + k_2\tau}, \qquad \varphi_2(\tau) = \frac{-h_2\tau - (1 + h_2)}{-h_2\tau + (1 + h_2)} \tag{2.9}$$

One may show that Equation (2.7) possesses one, and only one, nonzero root for the values of the parameters h_1 and h_2 satisfying conditions (2.5) and for these values $\phi_2(\tau) < 1$.

Values of the parameters a, h_1 and h_2 satisfying the equalities (2.6) to (2.9) will be bifurcate. For these values there is a double semistable limit cycle in the upper phase half-plane encompassing the cylinder (Fig. 4). It is easy to see, considering the behavior of the curves $s_1 = s_1(s_2)$ and $s_3 = s_3(s_2)$, that with the increase of parameter a or decrease of parameter h_2 the semistable limit cycle breaks down into two limit cycles of different stability and disappears for reverse variation of the parameters.

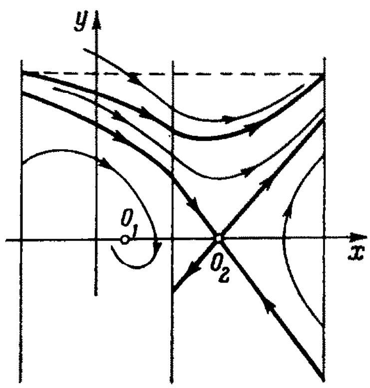

Fig. 4.

On the strength of the equalities (1.7) and (1.13) the conclusions regarding the quantity of limit cycles for the half-plane $y > 0$ and for the values of the parameters h_1 and h_2 will be valid for the half-plane $y < 0$ for values of the parameters $-h_1$ and $-h_2$. It is easy to see that the stability of the limit cycles in such a case will interchange. Bifurcate values of the parameters will be determined by the equalities (2.2) to (2.9) upon reversing the signs of h_1 and h_2 in these equalities.

3. Limit cycles not encompassing the cylinder. Let us consider the complex transformation $\Pi_2 = \Pi_2^{(1)}\Pi_2^{(2)}$. It is defined for the case $|h_1| < 1$. To the immovable points of this transformation correspond the limit cycles not encompassing the cylinder and intersecting only the line $x = \pi/2$. On the strength of the relations (1.18) and (1.20) we conclude that the transformation Π_2 can have immovable points only for different signs of the parameters h_1 and h_2. In order to find the immovable points it is necessary to solve the system

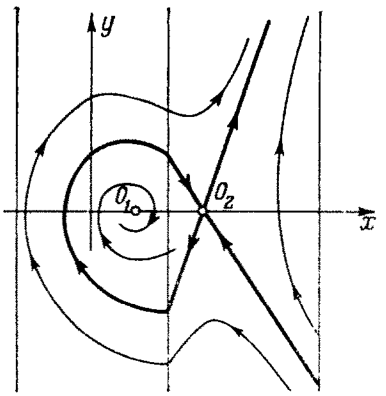

Fig. 5.

$$r_2(\tau_2) = r_2(\theta_2),\ s_2(\tau_2) = s_2(\theta_2) \quad (3.1)$$

(See (1.13) and (1.20)). Utilizing the relations (1.13) to (1.17), (1.20), as well as the inequality (1.19) and the Koenings theorem, it is not difficult to prove that the system (3.1) can have no more than one solution to which corresponds a stable (unstable) limit cycle for $h_1 < 0$ ($h_1 > 0$). The values of parameters satisfying the equality

$$F(h_1, h_2) = \frac{1}{2}\ln\frac{(\omega_2 + h_2)(\omega_2 + h_1)}{(\omega_2 - h_2)(\omega_2 - h_1)} + k_1 \tan^{-1}\frac{\omega_1\omega_2}{-h_1h_2} + k_1\pi = 0 \quad (3.2)$$

and the inequality

$$s_2(\tau_0) > (h_2 + \omega_2)(\pi/2 - a),\ r_2(\tau_0') > (-h_2 + \omega_2)(\pi/2 - a) \quad (3.3)$$

will be bifurcate.

Here $s_2(r_0)$ and $r_2(r_0')$ are determined from the equations of correspondence functions for transformations $L^{(1)}$ and $L_-^{(1)}$, while r_0 and r_0' are values of the parameters r and r' for which $s_1(r_0) = r_1(r_0') = 0$.

When conditions (3.2) and (3.3) are satisfied on the phase plane, the separatrix passes from saddle to saddle not encompassing the cylinder and intersecting only the line $x = \pi/2$ (Fig. 5). One can prove that for $h_1 > 0$ ($h_1 < 0$) the system (3.1) has no solutions if the expression $F(h_1, h_2) > 0$ ($F(h_1, h_2) < 0$). Consequently, there are no such limit cycles on the phase plane in these cases. If, however, for $h_1 > 0$ ($h_1 < 0$) the expression $F(h_1, h_2) < 0$ ($F(h_1, h_2) > 0$) and either one or both inequalities (3.3) are satisfied, then there is on the phase plane a limit cycle of the type considered.

Such limit cycles may not exist when both conditions of (3.3) are violated; however, then there will exist a limit cycle not encompassing the cylinder and which intersects both lines $x = \pi/2$ and $x = -\pi/2$. Such limit cycles evidently are determined by the immovable points of the complex transformation $\Pi = L^{(1)}\Pi_2{}^{(2)}L_-^{(1)}\Pi_3{}^{(2)}$ and exist only when h_1 and h_2 have different signs (see (1.18), (1.20), (1.21)). In order to find them it is necessary to solve a system of four equations with four unknowns (Fig. 1)

$$s_1(\tau) = s_3(\theta_3), \qquad s_2(\tau) = s_2(\theta_2), \qquad r_2(\tau') = r_2(\theta_2), \qquad \tau_1(\tau') = r_3(\theta_3)$$

(see (1.3), (1.7), (1.15), (1.20) and (1.21)).

Let point O_1 be a focus, i.e. $|h_1| < 1$, then the last system in both cases when $h_1 < 0$, $h_2 > 0$ and $h_1 > 0$, $h_2 < 0$ is reduced after transformations to the following system of two equations with two unknowns:

$$b = \tfrac{1}{2}\ln[u^2(\theta_2) + 2u(\theta_2)h_1 + 1] - \tfrac{1}{2}\ln[v^2(\theta_3) - 2v(\theta_3)h_1 + 1] + k_1\tau(\theta_2, \theta_3) \equiv F_1(\theta_2, \theta_3) \tag{3.4}$$

$$b = \tfrac{1}{2}\ln[v^2(\theta_2) - 2v(\theta_2)h_1 + 1] - \tfrac{1}{2}\ln[u^2(\theta_3) + 2u(\theta_3)h_1 + 1] - k_1\tau'(\theta_2, \theta_3) \equiv F_2(\theta_2, \theta_3)$$

Here in the case $h_1 < 0$, $h_2 > 0$ ($h_1 > 0$, $h_2 < 0$) the first equation corresponds to the upper (lower) phase half-plane, while the second one corresponds to the lower (upper) phase half-plane; quantities b, u and v are determined from (2.1) and (1.15), while the functions $\tau(\theta_2, \theta_3)$ and $\tau'(\theta_2, \theta_3)$ have the following sense:

$$\tau(\theta_2, \theta_3) \equiv \tan^{-1}\frac{\omega_1[u(\theta_2) + v(\theta_3)]}{q(\theta_2, \theta_3)} \equiv \varphi_1(\theta_2, \theta_3) \qquad \text{for } q \geqslant 0 \tag{3.5}$$

$$\tau(\theta_2, \theta_3) \equiv \pi - \tan^{-1}\frac{\omega_1[u(\theta_2) + v(\theta_3)]}{-q(\theta_2, \theta_3)} \equiv \varphi_2(\theta_2, \theta_3) \qquad \text{for } q \leqslant 0 \tag{3.6}$$

$$\tau'(\theta_2, \theta_3) \equiv \varphi_1(\theta_3, \theta_2) \quad \text{for } q(\theta_3, \theta_2) \geqslant 0 \tag{3.7}$$

$$\tau'(\theta_2, \theta_3) \equiv \varphi_2(\theta_3, \theta_2) \quad \text{for } q(\theta_3, \theta_2) \leqslant 0 \tag{3.8}$$

$$q(\theta_2, \theta_3) \equiv u(\theta_2)v(\theta_3) - h_1[u(\theta_2) - v(\theta_3)] - 1 \tag{3.9}$$

Utilizing Expressions (1.4), (1.7), (1.17), (1.20) and (1.21) as well as the inequality (1.19), one can prove that the system (3.4) possesses no more than one solution to which corresponds a stable (unstable) limit cycle for values of the parameters $h_1 < 0$, $h_2 > 0$ ($h_1 > 0$, $h_2 < 0$).

Let us establish the conditions for which the system (3.4) has a solution. Consider the equations

$$z = F_1(\theta_2, \theta_3), \qquad z = F_2(\theta_2, \theta_3) \tag{3.10}$$

where the functions $F_1(\theta_2, \theta_3)$ and $F_2(\theta_2, \theta_3)$ are determined by the equalities (3.4) as equations of two families of curves dependent on the parameter θ_3. The satisfaction of the equality $F_1(\infty, \theta_3) - F_2(\infty, \theta_3) > 0$ is a necessary condition for intersection of curves (3.10) corresponding to the same value of the parameter θ_3. In satisfying the above conditions the sufficient condition for the intersection of the curves is the fulfillment of the equality $F_1(0, \theta_3) - F_2(0, \theta_3) \leqslant 0$. It is not difficult to check the validity of these statements by considering partial derivatives of the function (3.10) and keeping in mind that in view of (1.19) the inequalities $\partial F_1/\partial\theta_2 > \partial F_2/\partial\theta_2$, $\partial F_1/\partial\theta_3 > \partial F_2/\partial\theta_3$ are satisfied for all values of θ_2 and θ_3.

From the above one can conclude: the system (3.10) has no solution if the expression $F_1(\infty, \infty) < 0$ for $h_1 < 0$, $h_2 > 0$ (respectively $F_1(\infty, \infty) > 0$ for $h_1 > 0$, $h_2 < 0$); if, however

$$F_1(\infty, \infty) = \frac{1}{2} \ln \frac{(\omega_2 + h_2)(\omega_2 + h_1)}{(\omega_2 - h_2)(\omega_2 - h_1)} + k_1 \tan^{-1} \frac{\omega_1\omega_2}{-h_1h_2} = 0 \qquad (3.11)$$

then the system (3.4) has a unique solution $\theta_2 = \theta_3 = \infty$ only for $a = 0$. On the phase plane the separatrix passes from saddle to saddle encompassing the cylinder in the lower as well as the upper phase half-plane (Fig. 6). Furthermore, if for $h_1 < 0$, $h_2 > 0$

$$F_1(\infty, \infty) > 0$$

$$F_1(\infty, 0) - F_2(\infty, 0) = \frac{1}{2} \ln \frac{(\omega_2 + h_2)(\omega_2 + h_1)}{(\omega_2 - h_2)(\omega_2 - h_1)} + k_1 \tan^{-1} \frac{\omega_1\omega_2}{-h_1h_2} + k_1\pi \leqslant 0 \qquad (3.12)$$

and for $h_1 > 0$, $h_2 < 0$

$$F_1(\infty, \infty) < 0, \qquad F_1(\infty, 0) - F_2(\infty, 0) \geqslant 0$$

then there exists a bifurcate value of the parameter $\theta_3 = \theta_3{}^*$ defined by the equality

$$F_1(\infty, \theta_3{}^*) = F_2(\infty, \theta_3{}^*) \qquad (3.13)$$

such that for all values of the parameter a for which $b \leqslant F_1(\infty, \theta_3{}^*)$ the system (3.4) has a solution (the uniqueness of this solution was shown above). For values of a satisfying $b > F_1(\infty, \theta_3{}^*)$ the system (3.4) has no solution. Let

$$b = F_1(\infty, \theta_3{}^*) \qquad (3.14)$$

Values of the parameters satisfying (3.13) and (3.14) are bifurcate. Indeed, when the values of the parameters satisfy conditions (3.12) to (3.14), the separatrix on the phase plane passes from saddle to saddle not encompassing the cylinder and intersects both lines $x = \pi/2$, $x = -\pi/2$ (Fig. 7). From this it follows that the limit cycle is generated in this case from the separatrix passing from a saddle to saddle

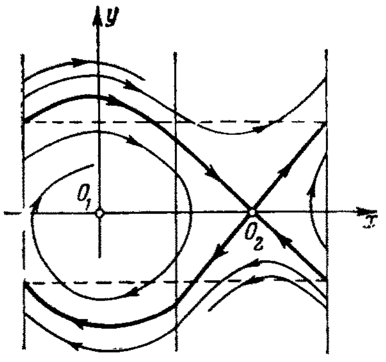

Fig. 6.

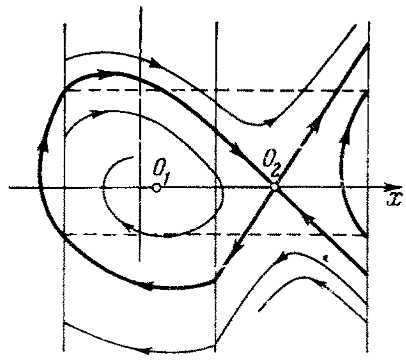

Fig. 7.

with decreasing parameter a.

Finally, it is not difficult to show that when the following conditions are satisfied

$$F_1(\infty, 0) - F_2(\infty, 0) > 0 \quad \text{for } h_1 < 0, \ h_2 > 0$$
$$F_1(\infty, 0) - F_2(\infty, 0) < 0 \quad \text{for } h_1 > 0, \ h_2 < 0$$

there exists in the phase plane a limit cycle not encompassing the cylinder and intersecting both lines $x = \pi/2$ and $x = -\pi/2$ if parameter a fulfills the condition $b < F_2(\theta_2{}^*, 0)$ in which $\theta_2 = \theta_2{}^*$ is determined from $F_1(\theta_2{}^*, 0) = F_2(\theta_2{}^*, \theta)$.

When the equality $b = F_2(\theta_2{}^*, 0)$ is fulfilled, there exists in the phase plane a limit cycle not encompassing the cylinder and touching the line $x = -\pi/2$. If, however, $b > F_2(\theta_2{}^*, 0)$, then there exists in the phase plane a limit cycle intersecting only the line $x = \pi/2$ (see p. 1528). Under the conditions (see (3.12))

$$F_1(\infty, 0) - F_2(\infty, 0) = 0 \tag{3.15}$$

$$b = \frac{1}{2} \ln (\omega_2 + h_2)(\omega_2 + h_1) + k_1 \tau \tag{3.16}$$

where
$$\tau = \tan^{-1} \frac{\omega_1(\omega_2 + h_2)}{-h_1(\omega_2 + h_2) - 1}$$

$$\text{for } -h_1(\omega_2 + h_2) \geqslant 1$$

$$\tau = \pi - \tan^{-1} \frac{\omega_1(\omega_2 + h_2)}{h_1(\omega_2 + h_2) + 1}$$

$$\text{for } -h_1(\omega_2 + h_2) \leqslant 1$$

the separatrix passes from saddle to saddle in the phase plane and does not encompass the cylinder but touches the line $x = -\pi/2$ (Fig. 8).

In the case $|h_1| \geqslant 1$ the system (3.4) can have a solution only for values of the parameters $\theta_2 > \theta^\circ$, $\theta_3 > \theta^\circ$, where θ° is determined from the equality $u(\theta^\circ) = h_1 + \omega_1$. At the same time the expressions $q(\theta_2, \theta_3)$ and $q(\theta_3, \theta_2)$ (see (3.9)) are positive and the functions $r(\theta_2, \theta_3)$ and $r'(\theta_2, \theta_3)$ in (3.4) are determined, respectively, by Formulas (3.5) and (3.6) if for $|h_1| > 1$ on the right-hand side of these formulas one substitutes $\tanh^{-1}$ for $\tan^{-1}$ and for $|h_1| = 1$ the sign of $\tan^{-1}$ is dropped.

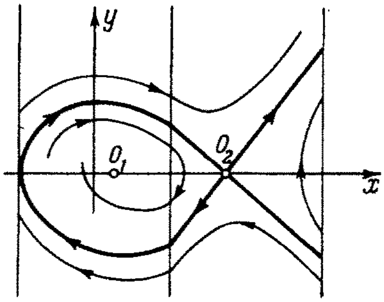

Fig. 8.

It can be shown that in the case $|h_1| \geqslant 1$, the system (3.4) has no solution of the left part of the equality

$$\frac{1}{2}\ln\frac{(\omega_2+h_2)(\omega_2+h_1)}{(\omega_2-h_2)(\omega_2-h_1)} + k_1\tanh^{-1}\frac{\omega_1\omega_2}{-h_1h_2} = 0 \tag{3.17}$$

is positive (negative) for values $h_1 < 0$, $h_2 > 0$ ($h_1 > 0$, $h_2 < 0$). If, however, the left part of the equality is negative (positive) for $h_1 < 0$, $h_2 > 0$ ($h_1 > 0$, $h_2 < 0$), then the system (3.4) has a solution when the parameter a satisfies the inequality $b \leqslant F_1(\infty, \theta_2^*)$ (see (3.13)) and has no solution in the opposite case. The bifurcate values of the parameters a, h_1, h_2, for which the separatrix passes from saddle not encompassing the cylinder and intersecting both lines $x = \pi/2$ and $x = -\pi/2$ satisfy, as well as for $|h_1| < 1$, the equalities (3.13) and (3.14) which have been changed as stated above for cases $|h_1| > 1$ and $|h_1| = 1$.

4. Division of phase space. We will consider division of b_1, h_1, h_2 parameter space, into rough regions by bifurcate surfaces where b is defined by (2.1).

From the presented investigation if follows that it is sufficient to consider division of space for $h_1 > 0$. Reflecting the obtained division symmetrically with respect to the axis b, we will obtain the division of space for values of $h_1 < 0$. At the same time all cycles of the system corresponding to the reflected point of space will attain opposite stability and, if to the point (h_1°, h_2°, b°) there corresponds a system having a limit cycle encompassing the cylinder in the upper (lower) phase half-plane, then to the reflected point ($-h_1^\circ$ $-h_2^\circ$ $-b^\circ$) there will correspond a system having a limit cycle encompassing the cylinder in the lower (respectively upper) phase half-plane.

Part of the space $h_1 > 0$ is divided into rough regions by the bifurcate surfaces *1, 1', 2, 3, 4*.

The surfaces *1* and *1'* correspond to the appearance of limit cycles from a separatrix passing from saddle to saddle and encompassing the cylinder in the upper, respectively, lower phase half-planes.

The surfaces *2, 3* and *4* correspond to the appearance of limit cycles, respectively, from infinity, from compression of trajectories, and from the separatrix passing from saddle to saddle and not encompassing the cylinder.

The surface *1* is defined by the equalities (2,2), (2.3) and is intersected by the planes $h_1 = 0$ and $h_2 = 0$ along the lines Γ_1 and Γ_2 and the plane $b = 0$ along K. The equation of this line is given by the equalities (3.11) and (3.17).

The surface *1'* is symmetric to the surface *1* with respect to the axis b and intersects with it along the line K, while it intersects the plane $h_1 = 0$ along Γ_1' symmetric to Γ_1, with respect to the axis b.

The surface *2* is a bisectional surface (see (2.4)).

The surface *3* is defined by the equalities (2.6) to (2.9). To its points correspond the systems having a double semistable limit cycle. It is not difficult to show that the surface *3* intersects the plane $b = 0$ along the bisectrix $h_1 + h_2 = 0$, that it is located below the surface *1* and contacts the surface *1* along the line Γ_2, which is its intersection with surface $h_2 = 0$. It can be shown also that the value of b, determined from the equations for surfaces *1* and *3*, tends to infinity on the hyperbola $h_1{}^2 - h_2{}^2 = 1$.

The surface *4* consists of two parts which are in contact with each other along the line C_*, defined by the equalities (3.15), (3.16). One part of the surface *4* is projected into the region of plane $b = 0$, bounded between the line K and the line C which is a projection of line C_* on the plane $b = 0$. The second part is located above the line C_* and is projected into the line C, i.e. it coincides with the cylindrical surface $F(h_1, h_2) = 0$ (see (3.2) and (3.15)). To the points of the first (second) part of surface *4* correspond systems whose separatrices are shown located in Fig. 7 (Fig. 5).

The location of separatrices of systems corresponding to points on line C_* is shown in Fig. 8.

The surface *4* intersects with the surfaces *1* and *1'* and the plane

$b = 0$ along the line K. To the points of this line correspond systems the location of whose separatrices is shown in Fig. 6.

The above-considered surfaces divide part of the space $h_1 > 0$ into seven regions. We will indicate the boundaries of each region, the number and the character of system cycles corresponding to the points of a given region.

Region (*1*) is bounded by part of plane $h_1 = 0$ below line Γ_1, by part of plane $b = 0$ located between axis $h_1 = 0$ and line $h_1 + h_2 = 0$, and by the planes *1* and *3* (there are no limit cycles).

Region (2) is bounded by part of plane $h_1 = 0$ above line Γ_1 and by the surfaces *1* and *2* (one stable limit cycle encompassing the cylinder in the upper half-plane).

Region (*3*) is bounded by the surfaces *1*, *2* and *3* (two limit cycles of different stability encompassing the cylinder in the upper phase half-plane).

Region (*4*) is bounded by the surfaces *1*, *2* and part of the plane $b = 0$ located between the line $h_1 + h_2 = 0$ and line K (one unstable limit cycle encompassing the cylinder in the upper phase half-plane, and one unstable limit cycle encompassing the cylinder in the lower phase half-plane).

Region (5) is bounded by the surfaces *1*, *2* and *4* (one unstable limit cycle encompassing the cylinder in the lower phase half-plane).

Region (6) is bounded by the surfaces *1'*, *4* and by a part of plane $h_1 = 0$ above line Γ_1' (one unstable limit cycle encompassing the cylinder in the lower phase half-plane, and one unstable limit cycle not encompassing the cylinder).

Region (7) is bounded by a part of the plane $h_1 = 0$ below line Γ_1', by part of plane $b = 0$ located between the axis $h_1 = 0$ and line K, and by the surface *1'* (one unstable limit cycle not encompassing the cylinder).

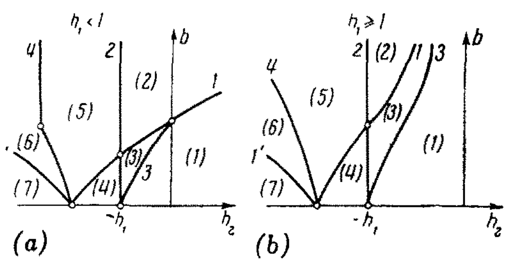

Fig. 9.

Schematic location of the above-enumerated regions is shown in Fig. 9*a* ($0 < h_1 < \text{const} < 1$) and in Fig. 9*b* ($h_1 = \text{const} \geqslant 1$).

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